$$KQ = \frac{+15 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}} - \beta}{16},$$

$$LQ = \frac{+15 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}} + \beta}{16},$$

$$LN = \sqrt{AL \cdot LQ} = \tfrac{1}{8}\sqrt{+34 + 2\sqrt{17} + 2\sqrt{34 + 2\sqrt{17}} - 2\beta'},$$

$$KM = \sqrt{AK \cdot KQ} = \tfrac{1}{8}\sqrt{+34 + 2\sqrt{17} + 2\sqrt{34 + 2\sqrt{17}} + 2\beta'},$$

$$[\beta' = \sqrt{2(\sqrt{17} - 3)(2\sqrt{17} - \sqrt{34 + 2\sqrt{17}})}; \quad \text{and} \quad \beta \times (-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}) = 4\cdot\beta'],$$

$$LN \times KM = \tfrac{4}{64}\sqrt{2(\sqrt{17} + 3)(2\sqrt{17} + \sqrt{34 - 2\sqrt{17}})} = \frac{4\alpha}{64},$$

$$\begin{aligned} NM^2 &= LK^2 + (KM - LN)^2 \\ &= 4\cdot EH^2 + KM^2 + LN^2 - \frac{8\alpha}{64} \\ &= \frac{+136 - 8\sqrt{17} + 8\sqrt{34 - 2\sqrt{17}} - 8\alpha}{64}, \end{aligned}$$

$$[\alpha' = \sqrt{2(\sqrt{17} + 3)(2\sqrt{17} - \sqrt{34 - 2\sqrt{17}})}; \quad \text{and} \quad 4\alpha = \alpha' \times (-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})],$$

$$\therefore \quad NM^2 = \frac{+136 - 8\sqrt{17} + 8\sqrt{34 - 2\sqrt{17}} - 2(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})\alpha'}{64}.$$

And, in the circle with radius equal to unity, NM represents the value $2\sin(2\pi/17) \times 1$; and therefore

$$\begin{aligned} 4\cos^2\frac{2\pi}{17} &= 4 - NM^2 \\ &= \frac{+120 + 8\sqrt{17} - 8\sqrt{34 - 2\sqrt{17}} + 2(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}})\alpha'}{64} \\ &= \left[\frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + \alpha'}{8}\right]^2 \end{aligned}$$

and

$$2\cos\frac{2\pi}{17} = \frac{-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + \sqrt{2(\sqrt{17} + 3)(2\sqrt{17} - \sqrt{34 - 2\sqrt{17}})}}{8}.$$

And this value of $2\cos(2\pi/17)$ will be found to agree with that given in Klein's *Famous Problems in Elementary Geometry* (Beman & Smith), though our α' is there written in a different form.

II. Historical Note by R. C. Archibald, Brown University.

This method of construction is due to H. W. Richmond, *Quarterly Journal of Mathematics*, Volume 26, 1893, pp. 206–207; and *Mathematische Annalen*, Volume 67, 1909, pp. 460–461. It is reproduced on page 34 of H. P. Hudson's *Ruler and Compasses*, London, 1916.

Various constructions of the regular polygon of seventeen sides were reviewed by R. Goldenring in his *Die elementargeometrischen Konstruktionen des regelmässigen Siebzehnecks* (Leipzig, 1915), but many omissions in this professedly complete survey were noted by the writer in the *Bulletin of the American Mathematical Society*, vol. 22, 239-246. The first solution in an English publication, given by Lowry in 1819,[1] was reproduced in this *Monthly*[2] in 1899 and 1914. Other solutions and historical notes are set forth in the articles printed above, pages 322–326.

2767 [1919, 171]. Proposed by W. W. JOHNSON, U. S. Naval Academy.

Let the complex quantities p, q, and r satisfy the relation $p^2 + q^2 + r^2 = 0$; prove that the corresponding vectors OP, OQ, and OR are such that if any two of them are taken as conjugate semi-diameters of an ellipse, the third lies on the minor axis, and its length is the distance from the center to either focus.

Solution by A. Pelletier, Montreal, Can.

Let $(x^2/a^2) + (y^2/b^2) = 1$, be the equation of the ellipse having OP and OQ for conjugate semi-diameters ($2a$ and $2b$ being the axes, and $a \geqq b$). If α, α', α'' are the respective arguments of

[1] *The Mathematical Repository*, new series, vol. 4, p. 160; Lowry's proof occupies pages 160-168.

[2] Volume 6, p. 239 and volume 21, p. 252.

OP, OQ, and OR, we have

$$OP^2(\cos 2\alpha + i \sin 2\alpha) + OQ^2(\cos 2\alpha' + i \sin 2\alpha') = - OR^2(\cos 2\alpha'' + i \sin 2\alpha''),$$

from datum; hence,

$$OP^2 \cos 2\alpha + OQ^2 \cos 2\alpha' = - OR^2 \cos 2\alpha'' \tag{1}$$

and

$$OP^2 \sin 2\alpha + OQ^2 \sin 2\alpha' = - OR^2 \sin 2\alpha''. \tag{2}$$

Now P and Q being points on the ellipse, we have from known properties,

$$OP^2 \cos^2 \alpha + OQ^2 \cos^2 \alpha' = a^2, \qquad OP^2 \sin^2 \alpha + OQ^2 \sin^2 \alpha' = b^2;$$

hence, $OP^2 \cos 2\alpha + OQ^2 \cos 2\alpha' = a^2 - b^2$, and (1) becomes

$$- OR^2 \cos 2\alpha'' = a^2 - b^2. \tag{3}$$

Also, from known properties concerning the ends of conjugate diameters,

$$OP^2 \sin 2\alpha = - OQ^2 \sin 2\alpha';$$

hence, (2) becomes

$$- OR^2 \sin 2\alpha'' = 0. \tag{4}$$

It follows from (3) and (4), that $2\alpha'' = 180°$ or $540°$, and $OR^2 = a^2 - b^2$, that is, $OR = \sqrt{a^2 - b^2}$, the distance from the center to focus, and $\alpha'' = 90°$ or $270°$, which shows that OR lies on the minor axis.

Also solved by H. Halperin, A. M. Harding, and H. L. Olson.

2780 [1919, 311]. Proposed by ELMER LATSHAW, West Philadelphia, Pa.

A quadrilateral whose sides are a, $2a$, $3a$, $4a$ is inscribed in a circle. Find the radius of the circle.

I. Solution by H. S. Uhler, Yale University.

The interest in this problem may be enhanced by giving a perfectly general solution. Let the sides of any convex inscriptible quadrilateral be denoted by a_1, a_2, a_3, a_4. A diagonal c may be drawn dividing the quadrilateral into two non-overlapping triangles the sides of which are a_1, a_2, c and a_3, a_4, c, respectively. If the angle between a_1 and a_2 be symbolized by C, the angle between a_3 and a_4 must be $180° - C$. Accordingly

$$c^2 = a_1^2 + a_2^2 - 2a_1a_2 \cos C,$$

$$c^2 = a_3^2 + a_4^2 + 2a_3a_4 \cos C.$$

Eliminating $2\cos C$ we find

$$c^2 = \frac{(a_1a_3 + a_2a_4)(a_2a_3 + a_4a_1)}{a_1a_2 + a_3a_4}. \tag{1}$$

The area of a plane triangle having the sides a_1, a_2, c is given by either member of the following equation

$$\frac{a_1a_2c}{4R} = \sqrt{s(s - a_1)(s - a_2)(s - c)}, \tag{2}$$

where $2s = a_1 + a_2 + c$, and R denotes the radius of the circumscribed circle.

Substituting the trinomial value of s in equation (2) we obtain

$$\frac{a_1a_2c}{R} = \sqrt{[(a_1 + a_2)^2 - c^2][c^2 - (a_1 - a_2)^2]}. \tag{3}$$

Replacing c in equation (3) by expression (1) we eventually find that

$$R = \frac{\sqrt{(a_1a_2 + a_3a_4)(a_1a_3 + a_4a_2)(a_2a_3 + a_4a_1)}}{\sqrt{(a_2 + a_3 + a_4 - a_1)(a_3 + a_4 + a_1 - a_2)(a_4 + a_1 + a_2 - a_3)(a_1 + a_2 + a_3 - a_4)}}, \tag{4}$$

or

$$R = \frac{1}{4K}\sqrt{(a_1a_2 + a_3a_4)(a_1a_3 + a_4a_2)(a_2a_3 + a_4a_1)}. \tag{5}$$

where if $2S = a_1 + a_2 + a_3 + a_4$, $K = \sqrt{(S - a_1)(S - a_2)(S - a_3)(S - a_4)}$ = area of quadrilateral.